Pointwise ergodic theorems for semigroup actions or: a sequence of increasingly more abstract/ harder to state results

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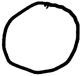
Part 0: Overview

Definitions and notation

- (X, μ) is a standard probability space.  w/ λ
- T is a Borel transformation on X. Irrational rotation by α
- Γ is a countable (semi)group acting on X in a Borel way.
- T is probability measure preserving (pmp) if... $\mu(A) = \mu(T^{-1}A)$

  intuitively, weight (x) = weight $(T^{-1}x)$
- $A \subseteq X$ is invariant if... $A = T^{-1}A$

  equiv., A is a union of orbits
- T is ergodic if...

  every invariant, measurable $A \subseteq X$ is null or conull.

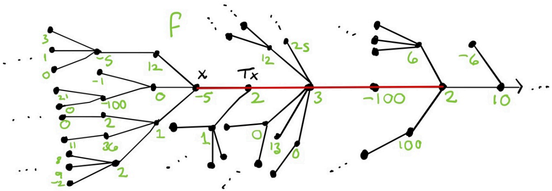
Classical pointwise ergodic theorem for a transformation

Theorem (Classical pointwise ergodic theorem, Birkhoff 1931)

A pmp transformation $T : X \to X$ is ergodic if and only if for each $f \in L^1(X, \mu)$ and for a.e. $x \in X$,

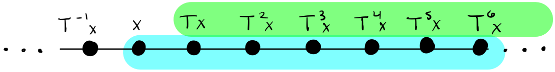
$$\lim_{n\to\infty} \left(\text{average of } f \text{ over } \left\{ x, Tx, \ldots, T^{n-1}x \right\} \right) = \int_X f \, d\mu.$$

$$\frac{1}{n} \sum_{i<n} f(T^i x)$$



Amenable groups

- In the classical pointwise ergodic theorem for pmp actions $\mathbb{Z} \curvearrowright (X, \mu)$, the averages are taken over $\{0, 1, \ldots, n\} \cdot x$.



sequence of finite $F_n \subseteq \Gamma$

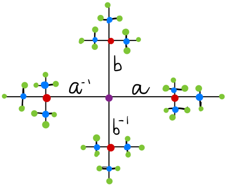
Averages of f over $F_n \cdot x$ $\xrightarrow{?}$ $\int f \, d\mu$

- The sequence $F_n := \{0, 1, \ldots, n\} \subseteq \mathbb{Z}$ works because it is a Følner sequence (the sets have small boundary relative to their size).

- This holds more generally for amenable groups:
  - **Lindenstraus** (2001)**.** The pointwise ergodic theorem holds for all pmp actions of Λ amenable groups along tempered Følner sequences.

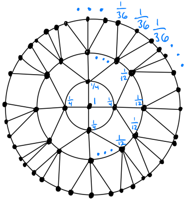
Nonamenable groups

- Take, for example, the free group $\mathbb{F}_r$ on $r \geqslant 2$ generators.
- Its finite subsets have large boundary, for example, the balls B_n:



- We *make* the boundary small by assigning weights, so each sphere S_n receives total weight 1.
- In other words, we take a non-backtracking simple random walk on $\mathbb{F}_r$.

Let $\mathfrak{m}_u$ denote the **uniform** distribution on each sphere S_n.



A pointwise ergodic theorem for free groups...

along balls with uniform weights

Theorem (Grigorchuk 1987; Nevo 1994)

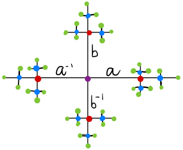
Let $\mathfrak{m}_u$ be the measure on $\mathbb{F}_r$, $r < \infty$, that is the uniform probability distribution on each sphere (non-backtracking simple random walk).

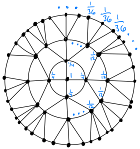
A pmp action $\mathbb{F}_r \curvearrowright (X, \mu)$ is ergodic iff for any $f \in L^1(X, \mu)$, for a.e. $x \in X$,

$$\frac{1}{n+1} \sum_{\gamma \in B_n} f(\gamma \cdot x)\, \mathfrak{m}_u(\gamma)$$

$$\mathfrak{m}_u\text{-weighted average of } f \text{ over } B_n \cdot x \to \int_X f \, d\mu \text{ as } n \to \infty,$$

where B_n is the (closed) ball of radius n of the Cayley graph of $\mathbb{F}_r$.





A pointwise ergodic theorem for free groups...

along balls with Markov weights

Theorem (Bufetov 2000)

Let $\mathfrak{m}$ be a strictly irreducible stationary Markov measure on $S^{<\mathbb{N}}$, where S is the standard symmetric generating set of $\mathbb{F}_r$, $r < \infty$.

A pmp action $\mathbb{F}_r \curvearrowright (X, \mu)$ is ergodic iff for any $f \in L^1(X, \mu)$, for a.e. $x \in X$,

$$\mathfrak{m}\text{-weighted average of } f \text{ over } B_n \cdot x \to \int_X f \, d\mu \text{ as } n \to \infty,$$

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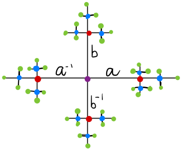
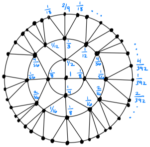


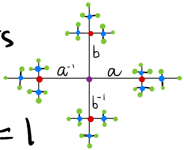


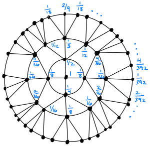
Part 1: pointwise ergodic theorem for the boundary action of the free groups

The boundary action $\mathbb{F}_r \curvearrowright \partial\mathbb{F}_r$

- $\mathbb{F}_r$: the free group on r generators, $r \leqslant \infty$

assign weights m so that $m(\text{sphere}) = 1$





- $\partial\mathbb{F}_r$: the space of infinite, reduced words in the generators of $\mathbb{F}_r$

$$[w] = \{x \in \partial\mathbb{F}_r \ : \ x \text{ starts with } w\}$$

$$\mu([w]) = m(w)$$

$\mathbb{F}_r \curvearrowright \partial\mathbb{F}_r$ by concatenation with cancellation

$$ab \cdot b^{-1}b^{-1}a^{-1}\cdots = ab^{-1}a^{-1}\cdots\cdots$$

Pointwise ergodic theorem for the boundary action

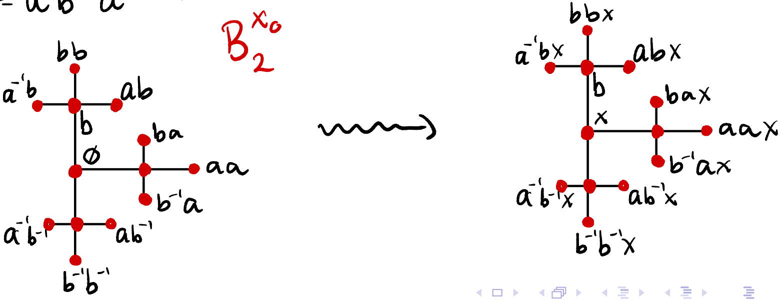
Corollary (Tserunyan–Z. 2020+)

If the boundary action is ergodic, then for each $f \in L^1(X, \mu)$ and for a.e. $x \in \partial\mathbb{F}_r$,

$$\lim_{n\to\infty}\left(\textit{weighted average of } f \textit{ over } \overbrace{\cancel{\left\{w \in B_n : w_{|w|} \neq x_1\right\} \cdot x}}^{B_n^{x_0}}\right) \cancel{=} \to \int_X f \, d\mu.$$

$S_x \cdot x$

$x = a b^{-1} a^{-1} \ldots\ldots$



Part 2: backward pointwise ergodic theorem for a transformation

The shift map on $\partial\mathbb{F}_r$

$$T : \partial\mathbb{F}_r \longrightarrow \partial\mathbb{F}_r$$

$$(x_n) \longmapsto (x_{n+1})$$

(forget first coordinate)

$$T^{-1}(x) = \{ax, bx, b^{-1}x\}$$

$$\overset{\shortparallel}{a^{\frown}x'} \quad = \{a_i x : a_i \neq x_0^{-1}\}$$

$$T^{-n}(x) = \{w \cdot x : |w| = n, w_{|w|} \neq x_0^{-1}\}$$

$$B_n^{x_0} = \bigcup_{i \leq n} T^{-i} x$$

Backward pointwise ergodic theorem for T

Corollary (Tserunyan–Z. 2020+)

A countable-to-one pmp transformation $T : X \to X$ is ergodic if and only if for each $f \in L^1(X, \mu)$ and for a.e. $x \in X$,

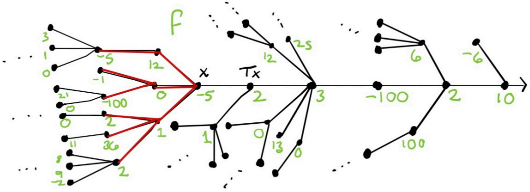
$$\lim_{n\to\infty} \left(\textit{weighted average of } f \textit{ over } \bigcup_{i\leqslant n} T^{-i}(x) \right) = \int_X f \, d\mu,$$

where the (relative) weights $\mathfrak{w}_x(y)$ come from the Radon–Nikodym cocycle.

$$\frac{1}{n+1} \sum_{y \in \bigcup_{i\leq n} T^{-i}x} f(y)\,\omega_x(y)$$

relative weights map $\omega_x(y)$

$$\sum_{y \in T^{-1}x} \omega_x(y) = 1$$



More general backward pointwise ergodic theorem for T

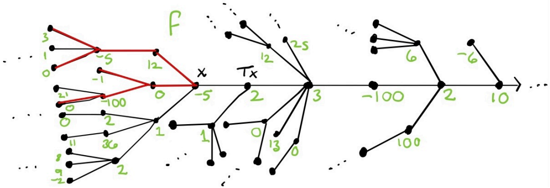
Theorem (Tserunyan–Z. 2020+)

A countable-to-one pmp transformation $T : X \to X$ is ergodic if and only if for each $f \in L^1(X, \mu)$ and for a.e. $x \in X$,

$$(\mathfrak{w}_x\text{-weighted average of } f \text{ over } S_x) \to \int_X f \, d\mu \quad \text{as} \quad \mathfrak{w}_x(S_x) \to \infty,$$

where S_x ranges over subtrees of the graph of T of finite height rooted at x and directed towards x.

$$\frac{1}{\omega_x(S_x)} \sum_{y \in S_x} f(y)\, \omega_x(y)$$



A pointwise ergodic theorem for free groups...

along balls with Markov weights

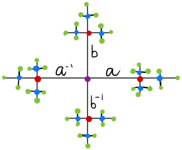
Theorem (Bufetov 2000)

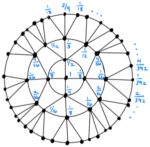
Let $\mathfrak{m}$ be a strictly irreducible stationary Markov measure on $S^{<\mathbb{N}}$, where S is the standard symmetric generating set of $\mathbb{F}_r$, $r < \infty$.

A pmp action $\mathbb{F}_r \curvearrowright (X, \mu)$ is ergodic iff for any $f \in L^1(X, \mu)$, for a.e. $x \in X$,

$$\mathfrak{m}\text{-weighted average of } f \text{ over } B_n \cdot x \to \int_X f \, d\mu \text{ as } n \to \infty,$$

where B_n is the (closed) ball of radius n of the Cayley graph of $\mathbb{F}_r$.





A pointwise ergodic theorem for free groups...

along trees with Markov weights

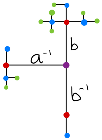
Theorem (Tserunyan–Z. 2020+)

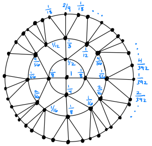
Let $\mathfrak{m}$ be a stationary Markov measure on $\mathbb{F}_r$, $r < \infty$, such that the measure of each $w \in \mathbb{F}_r$ is positive.

A pmp action $\mathbb{F}_r \curvearrowright (X, \mu)$ is ergodic iff for any $f \in L^1(X, \mu)$, for a.e. $x \in X$,

$$\mathfrak{m}\text{-weighted average of } f \text{ over } S \cdot x \to \int_X f \, d\mu \text{ as } \mathfrak{m}(S) \to \infty,$$

where S ranges over finite subtrees of the Cayley graph of $\mathbb{F}_r$ rooted at the identity.





From the free group action to a transformation

- $\langle S \rangle = \mathbb{F}_r \curvearrowright (X, \mu)$ finite, symmetric
- $T : X \times S^{\mathbb{N}} \to X \times S^{\mathbb{N}}$

$$(x, y) \longmapsto (y_0^{-1} x, s(y))$$

$\mu \times m$

$$T^{-n}(x, y) = \{ (w \cdot x, w \cdot y) : |w| = n \text{ no cancellation} \}$$

tree behind (x, y) from T $\longleftrightarrow$ tree in Cayley graph of $\mathbb{F}_r$

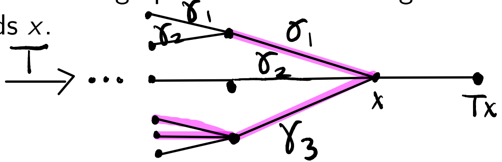
Part 3: forward pointwise ergodic theorem for a semigroup action

Generalizing the backward theorem

$$(\mathfrak{w}_x\text{-weighted average of } f \text{ over } S_x) \to \int_X f\, d\mu \quad \text{as} \quad \mathfrak{w}_x(S_x) \to \infty,$$

Borel (partial) right inverses of T

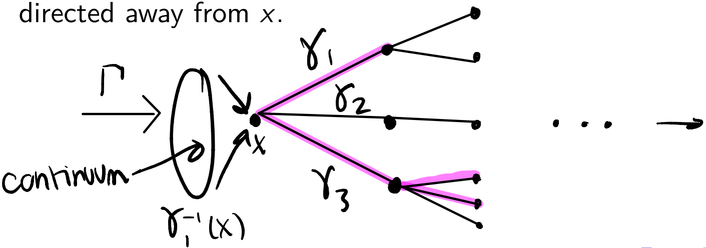
- Backward theorem: for a single transformation T, where S_x ranges over subtrees of the graph of T of finite height rooted at x and directed towards x.



$\sum_{\gamma} \omega_x(\gamma x) = 1$

$X = \bigsqcup \gamma \cdot X$

- More generally: for an action of a countable semigroup Γ, where S_x ranges over subtrees of the graph of Γ of finite height rooted at x and directed away from x.



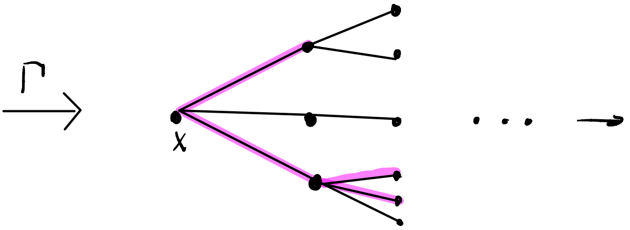
Forward pointwise ergodic theorem

Theorem (Tserunyan–Z. 2022+)

If Γ and $\mathfrak{w}$ are as below, and E_Γ is ergodic, then for each $f \in L^1(X, \mu)$ and for a.e. $x \in X$,

$$(\mathfrak{w}_x\text{-weighted average of } f \text{ over } S_x) \to \int_X f \, d\mu \quad \text{as} \quad \mathfrak{w}_x(S_x) \to \infty,$$

where S_x ranges over subtrees of the graph of T of finite height rooted at x and directed away from x.



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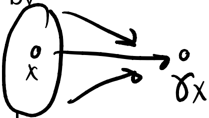
$$(\mathfrak{w}_x\text{-weighted average of } f \text{ over } S_x) \to \int_X f \, d\mu \quad \text{as} \quad \mathfrak{w}_x(S_x) \to \infty,$$

where S_x ranges over subtrees of the graph of T of finite height rooted at x and directed away from x.

Let $(w, x) \mapsto \mathfrak{w}_x(wx)$ be the relative weights map induced by

$$\mathfrak{w}_x(\gamma x) := \frac{d\mu|_{\gamma \cdot X}}{d\gamma^{\cancel{*}}_{*}\mu}(\gamma x).$$



Let $\Gamma = \{\gamma_i : i < N \leqslant \omega\}$ be a countable set of Borel partial transformations $\gamma_i : X \rightharpoonup X$. Assume that

- for each $n \in \mathbb{N}$, $X = \bigsqcup_{w \in \Gamma^n} w \cdot X$,
- for μ-a.e. $x \in X$, $\sum_{\gamma \in \Gamma} \mathfrak{w}_x(\gamma x) = 1$, and
- the equivalence relation E_Γ is null preserving.

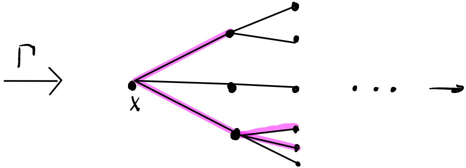
Pointwise ergodic theorem for a free semigroup action

Theorem (Tserunyan–Z. 2022+)

Let $\Delta = \langle \Gamma \rangle$ be a free semigroup, where $\Gamma = \{\gamma_i : i < N < \infty\}$, and let $\Delta \curvearrowright X$ be a (not necessarily free) pmp ergodic semigroup action. Let $\mathfrak{m}$ be a stationary Markov measure on $\Gamma^{<\mathbb{N}}$ with positive transition matrix such that the concatenation action $\Delta \curvearrowright \Gamma^{\mathbb{N}}$ is weakly mixing. Then for any $f \in L^1(X, \mu)$, for μ-a.e. $x \in X$, we have

$$\mathfrak{m}\text{-weighted average of } f \text{ over } S \cdot x \to \int_X f \, d\mu \text{ as } \mathfrak{m}(S) \to \infty,$$

where S ranges over finite subtrees of the Cayley graph of Δ rooted at the identity.



Thank you!